Hierarchy of time scales and quasitrapping in the N-atom micromaser

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Abstract

We study the dynamics of the reduced density matrix(RDM) of the field in the micromaser. The resonator is pumped by N-atomic clusters of two-level atoms. At each given instant there is only one cluster in the cavity. We find the conditions of the independent evolution of the matrix elements of RDM belonging to a (sub)diagonal of the RDM, i.e. conditions of the diagonal invariance for the case of pumping by N-atomic clusters. We analyze the spectrum of the evolution operator of the RDM and discover the existence of the quasitrapped states of the field mode. These states exist for a wide range of number of atoms in the cluster as well as for a broad range of relaxation rates. We discuss the hierarchy of dynamical processes in the micromaser and find an important property of the field states corresponding to the quasi-equilibrium: these states are close to either Fock states or to a superposition of the Fock states. A possibility to tune the distribution function of photon numbers is discussed.

PACS number 42.50
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Recent developments in the cavity electrodynamics [1] gave rise to the creation of a real physical device - micromaser which operates on highly excited Rydberg atoms pumped through a high-Q resonator [2]. Existing literature mostly focuses on the ideal (basic) model which is the so-called one-atom micromaser [3, 4, 5, 6]. This device is assumed to operate in such a way that no more than one atom excited with the probability 1 can be found in the cavity at each given instant of time. The basic model is justified by the following assumptions: the average velocity of injection R , and the time of interaction τ (in which a cluster passes through the resonator) are small. The rate of relaxation γ of the field is low, which means a high-Q resonator. The coupling constant g of the field mode interacting with internal degrees of freedom of the atom is sufficiently large. Trajectories are assumed to be quasiclassical. In more exact terms these assumptions can be recapitulated as

$$R\tau \ll 1,\ \tau\gamma \ll 1,\ g\tau \geq 1 \tag{1}$$

The micromaser operating on a periodic sequence of N-atomic clusters which are created by laser pulses in the gas of unexcited atoms is introduced in [7]. It is assumed that the size of the cluster is much less than the wavelength of the microwave radiation. Effects of finite cluster size [8] are comparable in magnitude with effects of inhomogeneous field at the edges of resonator. The latter observation was reported in Ref. [9]. In this Letter we study the one-cluster extension of the basic model. This means that we assume a point-like structure of N-atomic cluster (i.e. the finite size effects are not taken into account) as well as the fulfillment of the conditions Eq.(1). This formulation generalizes greatly the basic model while leaves intact the simplifying assumption that process of interaction of the cluster and the field (within the time interval τ) and the process of the field relaxation to the thermodynamic equilibrium (time interval $T \sim 1/R$) are separated in time. This latter assumption allows to factorize the evolution operator of the RDM (see Eq. (3) below) and greatly simplifies the analysis of the properties of evolution operator and the dynamics of RDM.

One-cluster model of the micromaser assumes that the N-particle Tavis-Cummings Hamiltonian [10]

$$H = H_0 + V = \omega \left(a^\dagger a + S_3 + \frac{N}{2} \right) + 2g \left(a^\dagger S_- + a S_+ \right) \tag{2}$$

is applicable. Here ω is the frequency of the quantum transition which is in exact resonance with the field mode. The collective spin variables $S_3, S_\pm$ are the generators of the $su(2)$ algebra, while $a^\dagger, a$ are the creation and annihilation operator of the field mode, $\hbar = 1$.
The operation of the micromaser for each cluster is divided into two time intervals: the interaction time τ and the relaxation time T. This means that the vector of the main diagonal $\rho^{(l)}$ of RDM satisfies the following equation

$$\begin{aligned} \rho^{(l+1)} &= S(N)\rho^{(l)} = Q(N_{ex})Sp_{at}\left(e^{-iH\tau}\rho_{at}\otimes\rho^{(l)}e^{iH\tau}\right) \\ &= Q(N_{ex})W(\tau)\rho^{(l)}. \end{aligned} \tag{3}$$

This difference equation connects the main diagonals of RDM taken at the instants when the l-th and $(l+1)$-th clusters enter the cavity. This allows to understand the number of passing clusters l as a discrete "time variable". Here $N_{ex} = R/\gamma$, and $Q(N_{ex})$ is the evolution operator of RDM at the relaxation stage, i.e. in the empty resonator [5]. The operator $W(\tau)$ describes the evolution of RDM at the stage of interaction of $(l+1)$-th cluster with the field, ρ_{at} is the density matrix of N-atomic cluster before it enters the resonator. The operation Sp_{at} means the trace with respect to atomic variables.
In our work we consider clusters of fully excited atoms, while the field is initially prepared in the state of the thermal equilibrium with the mean number of photons $n_b = 0.1$. In our forthcoming publication we will rigorously show that if there is an additive with respect to atoms and field integral of motion $[H, H_0] = 0$ and for unpolarized initial state of the cluster, then the dynamics of RDM is *diagonally invariant*. This important property of the evolution operator $W(\tau)$ means that each (sub)diagonal of RDM in the Fock basis evolves independently of other elements of RD matrix. In this Letter we concentrate on the dynamics of the main diagonal of RDM, i.e. on the number of photons probability distribution function. In the space of vectors with components $\rho_n^{(l)}$, the evolution operators $Q(N_{ex})$ can be represented as the following matrix

$$Q(N_{ex}) = \left(1 + \frac{L}{N_{ex}}\right)^{-1}, \tag{4}$$

where the operator L in the matrix form reads

$$L_{nm} = [-(n_b+1)n_b(n+1)]\delta_{nm} - (n_b+1)(n+1)\delta_{n+1,m} + n_b n\delta_{n-1,m}. \quad (5)$$

The matrix of the evolution operator $W(\tau)$ in the Fock basis is low-triangular. In the present Letter we analyze this matrix by numerical methods. The property of diagonal invariance simplifies greatly the analysis of RDM dynamics.

The vector of the main diagonal of RDM satisfies the following difference equation

$$\rho_n^{(l+1)} - \rho_n^{(l)} = J_n^{(l+1)} - J_n^{(l)}. \quad (6)$$

Here $J^{(l)}$ is the vector of the probability flux for the l-th passage. The components of this vector are

$$J_n^{(l)} = -\sum_{n'=0}^{n-1}\sum_{n''=n} S(N)_{n'n''}\rho_{n''}^{(l)} + \sum_{n'=n}\sum_{n''=0}^{n-1} S(N)_{n'n''}\rho_{n''}^{(l)} \quad (7)$$

This vector determines the rate of change (after one passage) of the sum of probabilities of the photon numbers in the interval of Fock numbers between $n = n_0$ and $n = n_1$. This rate is equal to the difference of fluxes through the chosen boundary values, viz

$$\sum_{n=n_0}^{n_1}\left(\rho_n^{(l+1)} - \rho_n^{(l)}\right) = J_{n_0}^{(l)} - J_{n_1+1}^{(l)} \quad (8)$$

The dependencies of the eigenvalues W_n on the number of photons are given in Figure 1 for the number of atoms in the cluster N=1,5,10. The interaction time is chosen as $g\tau = 1.355$. The eigenvalues W_n are positive and do not exceed 1. Their mutual positions are defined by the parameter τ and the number of atoms N. For the one-atom micromaser the so-called trapped states are known. These are the Fock states for the number of photons n corresponding to the eigenvalue $W_n = 1$ of the matrix $W(\tau)$. This number of photons fulfills the trapping condition

$$\sqrt{n+1} = \frac{\pi\chi}{g\tau} \quad (9)$$

where χ is an integer number. The trapped states do not decay in the absence of relaxation, and thus determine the dynamics of $\rho^{(l)}$ for large l. The recent

experimental realization of the trapped states was reported in [11]. The Figure 1 shows that in the multi-atomic case there are no trapped states. There are however a few eigenvalues which are close to 1. The corresponding eigenvectors in the space of the number of photons are localized around the numbers n for which $W_n \approx 1$. Such long-living vectors is natural to call *quasitrapped states*.
The Figure 2 shows the spectrum $S(N)$ in ascending order. It is interesting to notice that the eigenvalues of the evolution operator tend to group around zero when the number of atoms in the cluster increases. In the hierarchy of dynamical processes in the micromaser the small eigenvalues are responsible for the rapid phase of the dynamics (with respect to the discrete time l). The quasitrapped states corresponding to the eigenvalues in the interval $[0.9, 1)$ are in turn responsible for the slow phase of dynamics. Probabilities of the states with corresponding photon numbers at certain stages of the field formation can be rather high. In Figure 2 we compare the spectrum of the evolution operator $S(N)$ for the cases with ($N_{ex} = 20$) and without relaxation. The Figure 2 shows in particular that for bigger N the spectrum of $S(N)$ is more stable towards the influence of relaxation. The relation Eq.(3) describes the transition of the diagonal elements of RDM to a stationary state. This transition process is determined by the pumping of the cavity field by passing clusters as well as by the relaxation of the field. The recent literature discusses mostly [3, 5] the properties of the stationary state, which can be achieved when a large number of clusters has gone through the cavity. This case corresponds to the asymptotic limit $l \to \infty$. In this work we concentrate on the properties of the transition process which, due to the existence of the quasitrapped states, are very interesting. The field rather rapidly "forgets" its initial state of the thermal equilibrium. The dynamics of the population of the Fock states shows instead the formation of long-living (with respect to the "time" l) quasi-equilibrium distributions. This is illustrated by the properties of spectrum of the evolution operator given in Figures 1,2. The existence of the eigenvalues close to 1 indicates considerable probabilities of the Fock states with photon numbers in the vicinities of the maxima. The small eigenvalues correspond to the sharp depletion of the corresponding Fock states. The Figure 1 shows that in the chosen interval of Fock numbers, $0 \leq n \leq 60$, and for $g\tau = 1.355$ there are three domains capturing considerable probabilities. These domains, which are natural to call the domains of *quasitrapping* are localized in the vicinities of the Fock

numbers $n = 5, 18, 40$ and contain almost all the probabilities. This means that they are getting populated at different "moments" of "time" l in relays: the next domain cannot get populated until the previous one is depleted. This relay of populations is illustrated in Figures 3,4,5. The Figure 3 show for $N = 10$ how the sums of probabilities of the Fock states change with l in the second ($14 \leq n \leq 24$) and the third ($39 \leq n \leq 49$) domains of quasitrapping. The rates of probability change are calculated through Eq. (8) i.e. as the flux differences through the boundaries of the chosen domains. The Figure 3 allows to identify the following stages of the l-dynamics: a period of accumulation of the probability which corresponds to the positive values of the probability rate as well as an extended in time (l) period of the negative probability rates. The lasting nature of the latter period indicates that the life-time of the quasitrapped states is considerable. The rate of decay of the second quasitrapped state ($n \approx 18$) is approximately the rate of accumulation in the third state ($n \approx 40$). This means that through a passage of a cluster the probability is almost fully relayed from the second quasitrapped state to the third. Since the dynamics of the decay of the second quasitrapped state is slow, so is the dynamics of the accumulation in the third state. In Figures 4 and 5 are given dependencies on l of total populations curves of the domains of quasitrapping. It is evident from Figures 4 and 5 that the sum of populations of two subsequent domains of quasitrapping is close to 1. This again manifests the full accumulation of the probabilities in the domains of quasitrapping as well as the relay of probabilities indicated above. The Figure 6 shows the distributions of the diagonal elements of the RDM taken at the l -moments of maximal probabilities of the Fock states in the corresponding domains of quasitrapping. As follows from Figure 6 it is possible to govern the vector of photon number distribution. This can be achieved by the variation of the number of atoms passed through the resonator. It is possible in particular to create states close to the Fock states localized at certain photon numbers. We can also report that a domain of the localization changes smoothly in accord with variations of parameters τ, N and N_{ex} . The dependence of the stationary field on these parameters for $N = 1$ was discussed in Refs. [3, 5, 6]. The possibility to engineer quantum states is actively studied in the recent literature [12].

Conclusions and discussion

The main result of our work is the discovered possibility to purposefully create in the cavity quasistable states close to Fock states. We analyzed the dynamics of the micromaser pumped by N-atomic clusters [7]. Our approach generalizes the basic model of the one atom micromaser [3] and can be experimentally realized. We assumed the point-like nature of N-atomic clusters. This assumption can easily be realized in practice when clusters are created in a gas flow by focused laser pulses in the light range. In this case the width of the beam is of order of few microns while the size of the cavity can be of order of few millimeters. In our work we have pointed out the conditions when the time evolution of a (sub)diagonal of the reduced density matrix is independent of the other elements of the density matrix. We have investigated the properties of the spectrum of the evolution operator (see Figures 1,2) and discussed their connection to the properties of the RDM dynamics. We have discussed the hierarchy of the time scales of the micromaser dynamics and have shown that the sectors of the spectrum around zero are responsible for rapid processes while the sectors close to 1 correspond to quasi-equilibrium. For the first time in the existing literature we have introduced an important notion of the *quasitrapped states*. The Figure 6 shows that these states are close to the Fock states. The domains in the Fock space corresponding to quasitrapping are rather narrow, their locations change smoothly with variations of the number of atoms in the cluster. This means that the overall picture of the dynamics is stable with respect to small variations of the number of atoms in a cluster. In our future work we plan to investigate this phenomenon in a greater detail as well as to study how the properties of the quasitrapped states depend on the choice of the initial density matrix of the N-atomic cluster.

References

[1] S. Haroche, D. Kleppner, Phys. Today, **42**(1), 24(1989).

[2] D. Meschede, H. Walther, G. Muller, Phys. Rev. Lett. **54**, 551(1985); G. Rempe, M. Scully, H. Walther, Physica Scripta, **34**, 5 (1991); G.M. Brune, J. Raimond, P. Goy, L. Davidovich, S. Haroche, Phys. Rev. Lett. **59**, 1899 (1987).

[3] P. Filipowicz, L. Javanainen, P. Meystre, Phys. Rev. A, **34**, 3077 (1986).

[4] P. Meystre, M. Sargent III. Elements of Quantum Optica, Springer-Verlag, Berlin, 1990.

[5] P. Elmfors, B. Lautrup, B. Shagerstam, Phys. Rev. **54**, 5171 (1996).

[6] P. Meystre, G. Rempe, H. Walther, Opt. Lett. **13**, 1078 (1988).

[7] G. D'Ariano, N. Sterpi, A. Zucchetti, Phys. Rev. Lett. **74**, 900 (1995).

[8] M. Orszag, R. Ramirez, J. Retamal, C. Saavedra, Phys. Rev. A **49**, 2933 (1994); L. Ladron, M. Orszag, R. Ramirez, Phys. Rev. A **55**, 2471 (1997); M. Kolobov, F. Haake, Phys. Rev. A **55**, 3033 (1997).

[9] C. Yang, K. An, Phys. Rev. A **55**, 4492 (1997); F. Gasagrande, A. Lulli, S. Ulrega, Phys. Rev. A **60**, 1582 (1999).

[10] M. Tavis, E. Cummings, Phys. Rev. **170**, 379 (1968); M. Sculle, G. Meyer, H. Walther, Phys. Rev. Lett. **76**, 4144 (1996); A. Rybin, G. Kastelewicz, J. Timonen, N. Bogoliubov, J. Phys. A: Math. And Gen. **31**, 4705 (1998).

[11] M. Weidinger, B.T.H. Varcoe, R. Heerlein, and H. Walther, Phys. Rev. Lett. **82**, 3795 (1999). Phys. Rev. Lett. **82**, 3795 (1999).

[12] K. Vogel, V. Akulin, and W. Schleich, Phys. Rev. Lett. **71**, 1816 (1993); Shi-Biao Zeng, Guang-Gan Guo, Phys. Lett. A, **244**, 512 (1998); A. Kozhekin, G. Kurizki, and B. Sherman, Phys. Rev. A, **54**, 3535 (1996).

Figure captions

Figure 1. The spectrum of $W(0,\tau)$ for $N = 1, 5, 10$ and $g\tau = 1.355$.

Figure 2. The spectrum of the evolution operator $S(N)$ in ascending order for $N = 1, 15$, $N_{ex} = 20, \infty$, and $g\tau = 1.355$.

Figure 3. The rates of change of integral probabilities of the Fock states in the second $14 \leq n \leq 24$ and the third $39 \leq n \leq 49$ quasitrapping domains for $N = 10$.

Figure 4. Integral probabilities of the Fock states in the second $14 \leq n \leq 24$ and the third $39 \leq n \leq 49$ quasitrapping domains for $N = 1$.

Figure 5. Integral probabilities of the Fock states in the second $14 \leq n \leq 24$ and the third $39 \leq n \leq 49$ quasitrapping domains for $N = 10$.

Figure 6. The photon number distributions of the diagonal elements of RDM at the l -moments of maximal probabilities of the Fock states in the second $14 \leq n \leq 24$ and the third $39 \leq n \leq 49$ domains of quasitrapping.

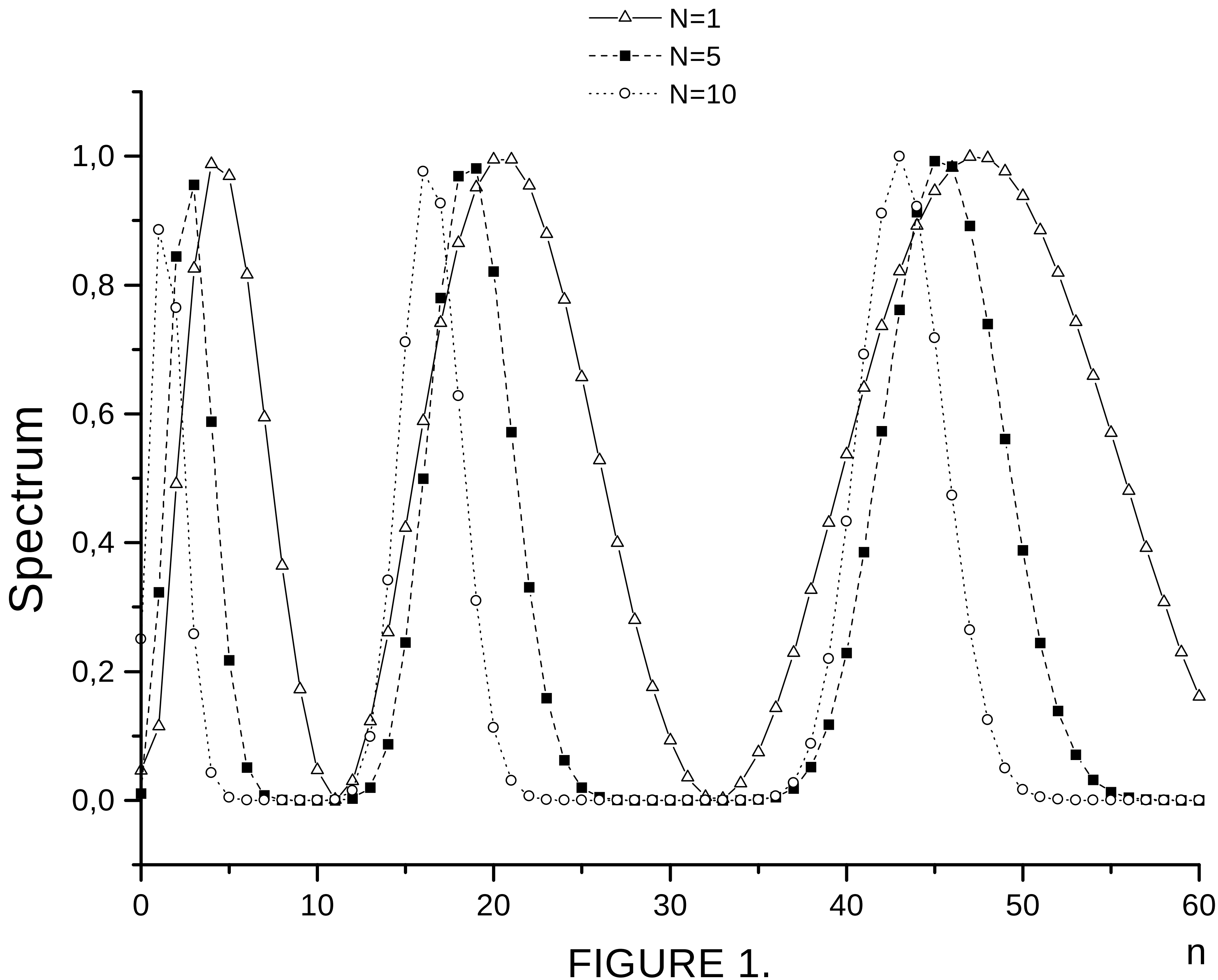
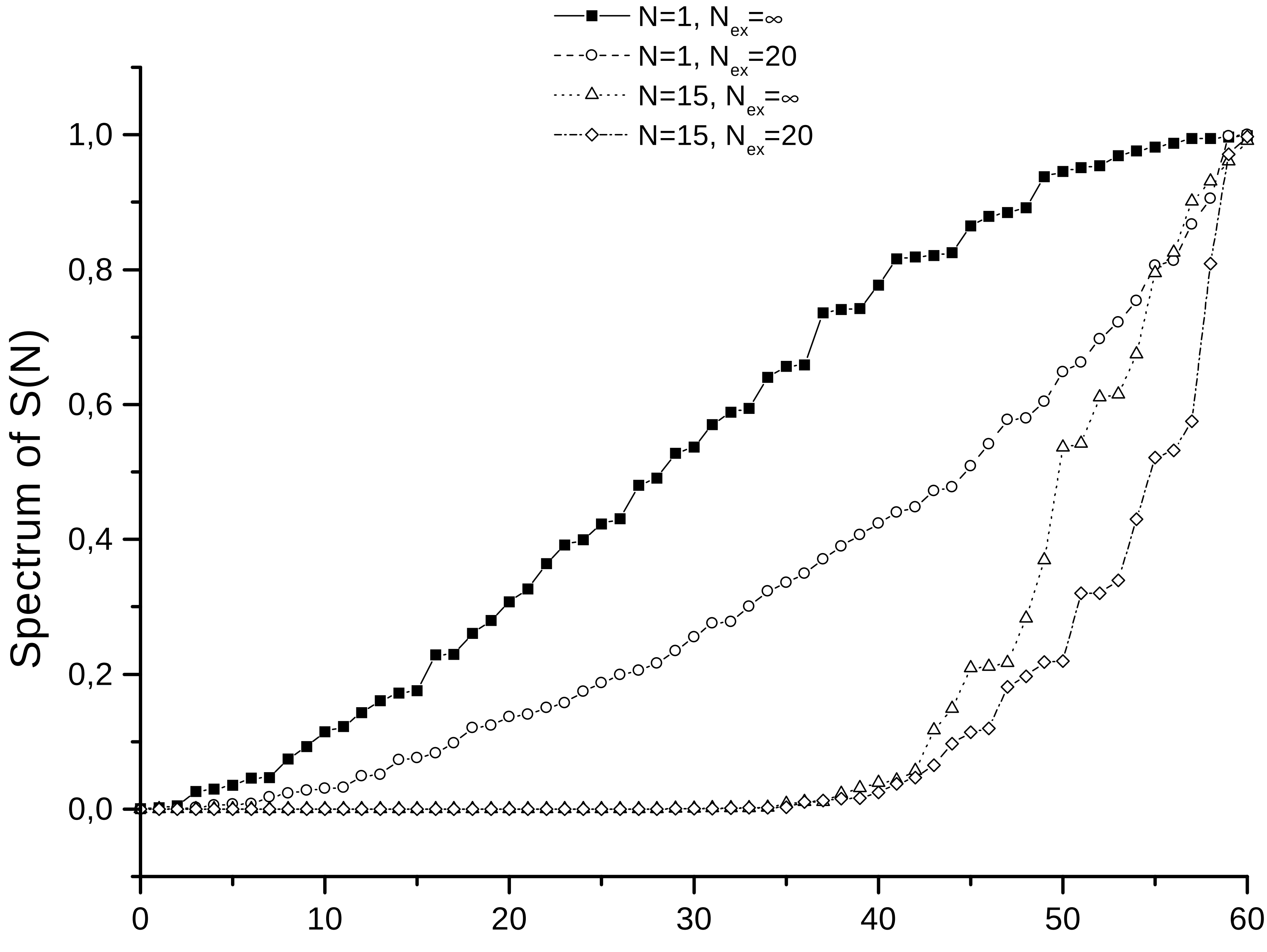
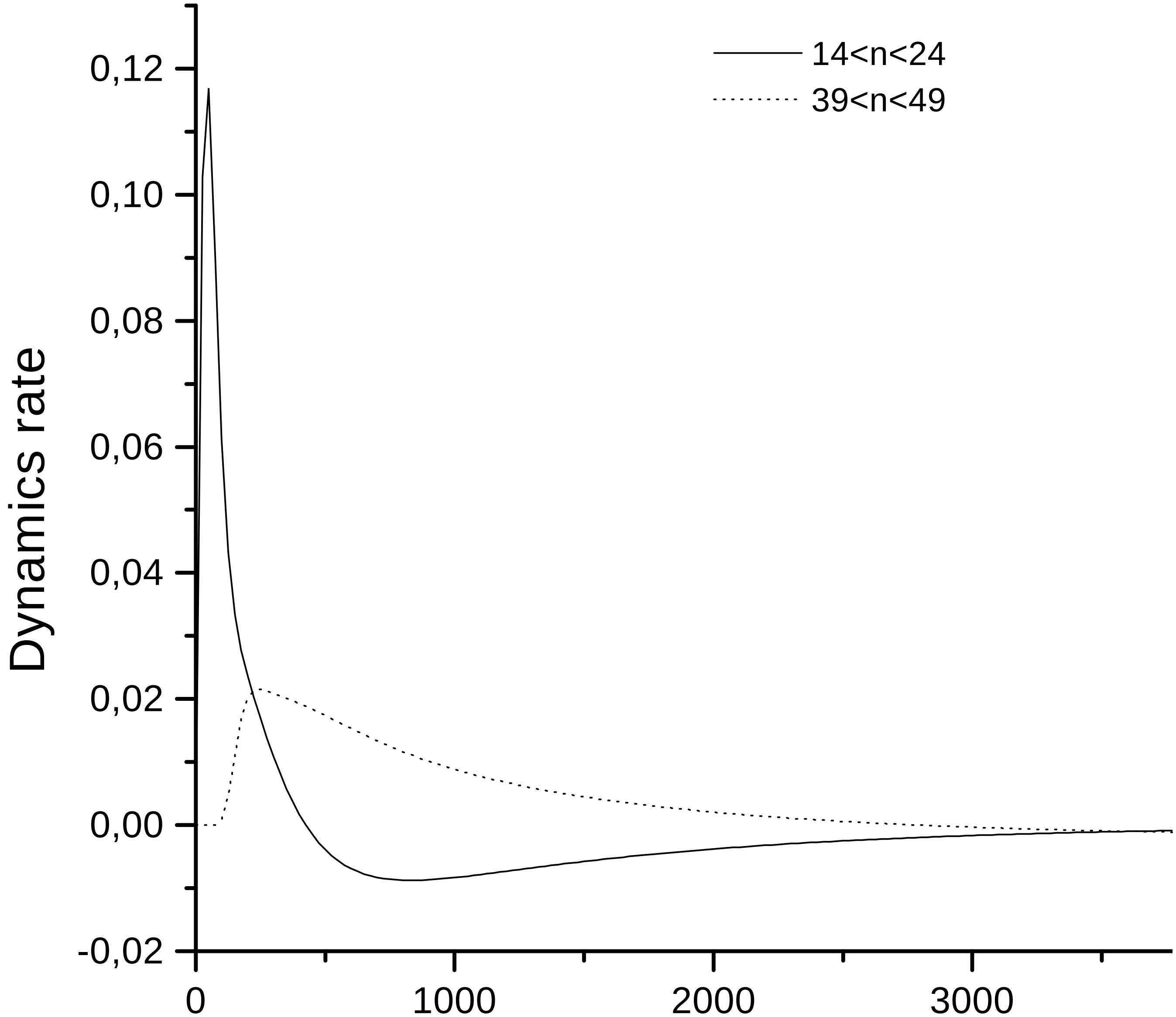


FIGURE 1.



N=1, N_{ex}=∞
N=1, N_{ex}=20
N=15, N_{ex}=∞
N=15, N_{ex}=20
Spectrum of S(N)
1,0
0,8
0,6
0,4
0,2
0,0
0
10
20
30
40
50
60

FIGURE 2.



14<n<24
39<n<49
Dynamics rate
0,12
0,10
0,08
0,06
0,04
0,02
0,00
-0,02
0
1000
2000
3000

FIGURE 3.

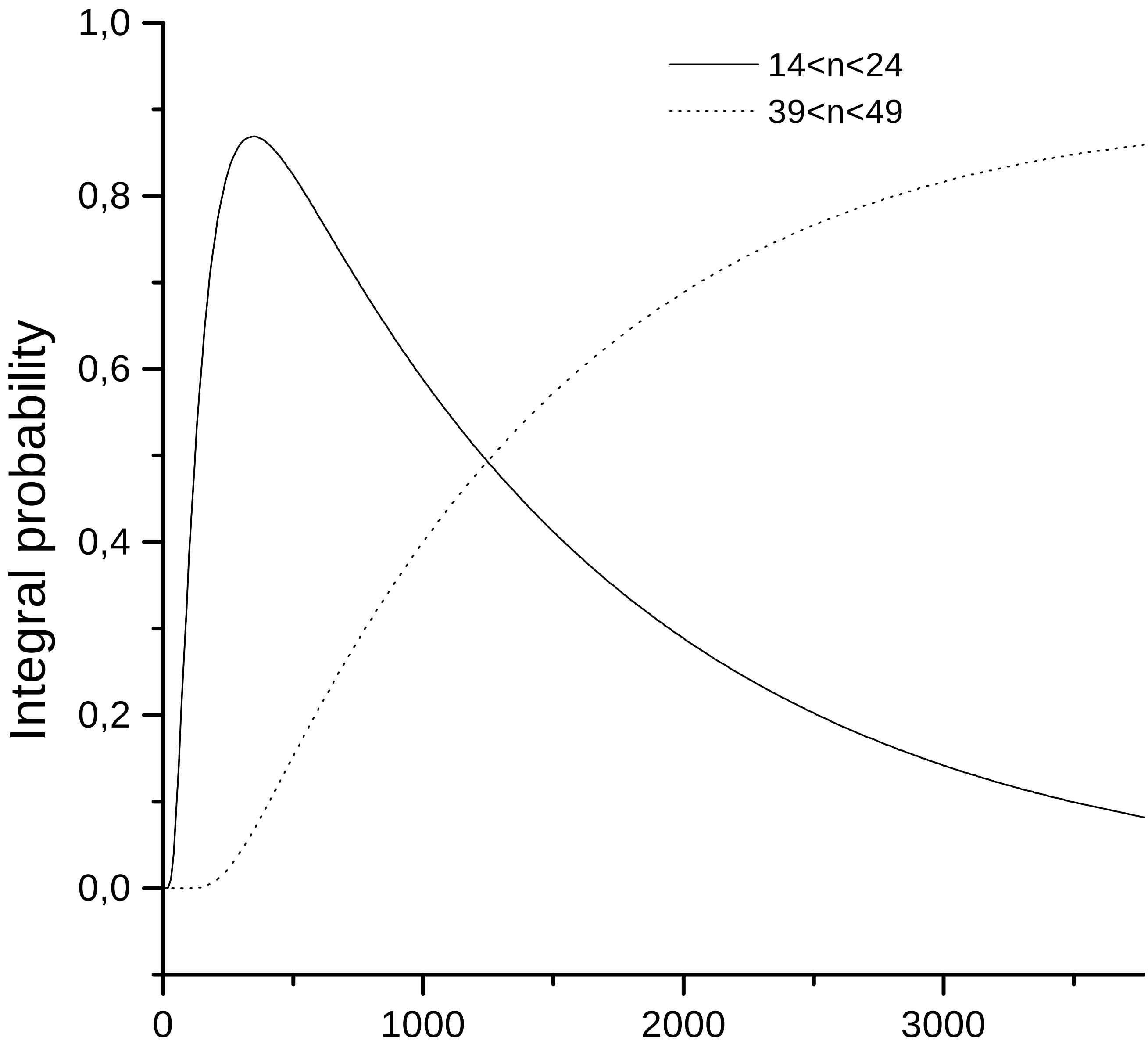

1,0
0,8
0,6
0,4
0,2
0,0
Integral probability
0
1000
2000
3000
14<n<24
39<n<49

FIGURE 4.

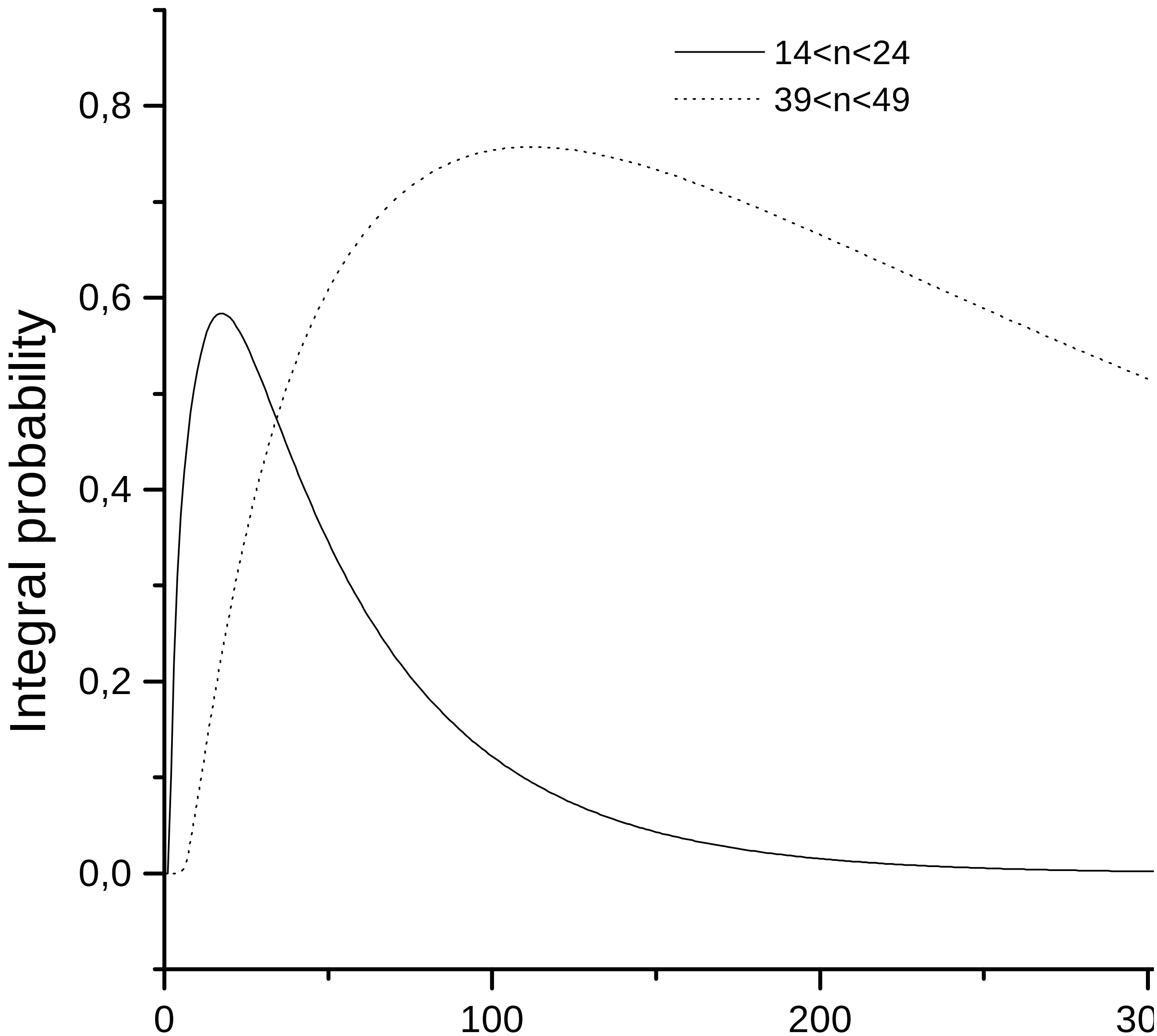


FIGURE 5.

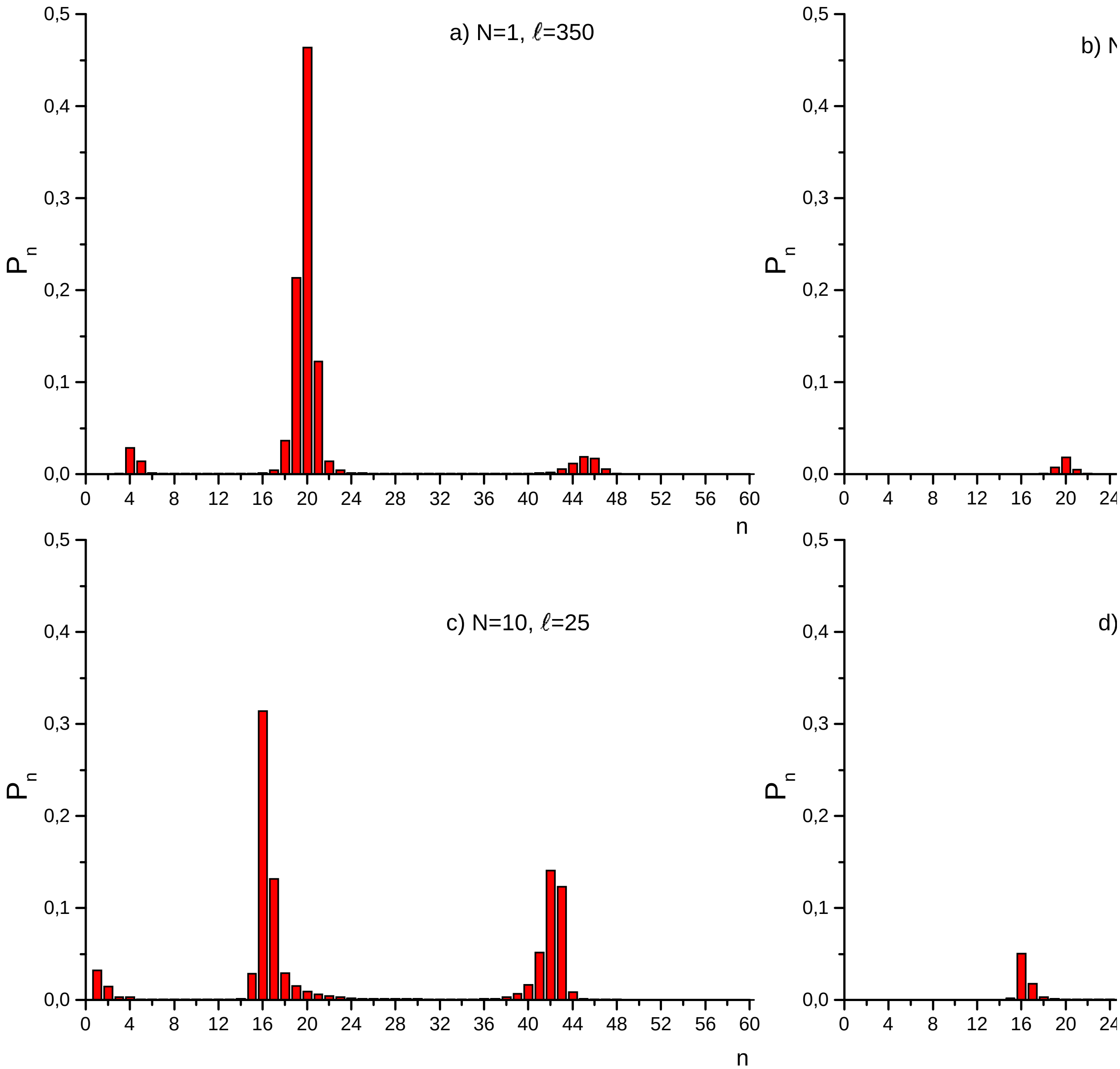


FIGURE 6.